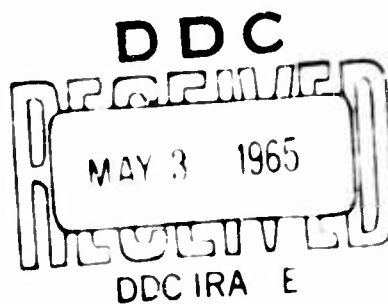FOUR—COLOR REDUCIBILITY OF PLANAR GRAPHS
CONTAINING SUBGRAPHS WITH FOUR-POINT BOUNDARIES

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FOUR-COLOR REDUCIBILITY OF PLANAR GRAPHS
CONTAINING SUBGRAPHS WITH FOUR-POINT BOUNDARIES

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It is well known that a planar graph containing a
proper subgraph with only three points in the boundary
between the subgraph and its complement is four-color
reducible—that is, if the subgraph and the complement
plus the boundary are each colorable with four colors,
then the original graph is four-colorable. A proof of
reducibility has been lacking for the case of subgraphs
whose boundaries contain four points. In the case of the
three-point boundary, if the graph is fully triangulated,
the three boundary points form a triangle, and they must
be assigned different colors: hence a common coloring
can be obtained by relabeling one of the two components
so that they match on the three points. If the boundary
contains four points, however, in the fully triangulated
case the boundary points will form a square. The two
components can be completed by connecting opposite corners
of the square, for example, z, y in Fig 1, in which case
z and y must be assigned different colors, but there is
no way of determining whether the two remaining corners
will have the same or different colors.

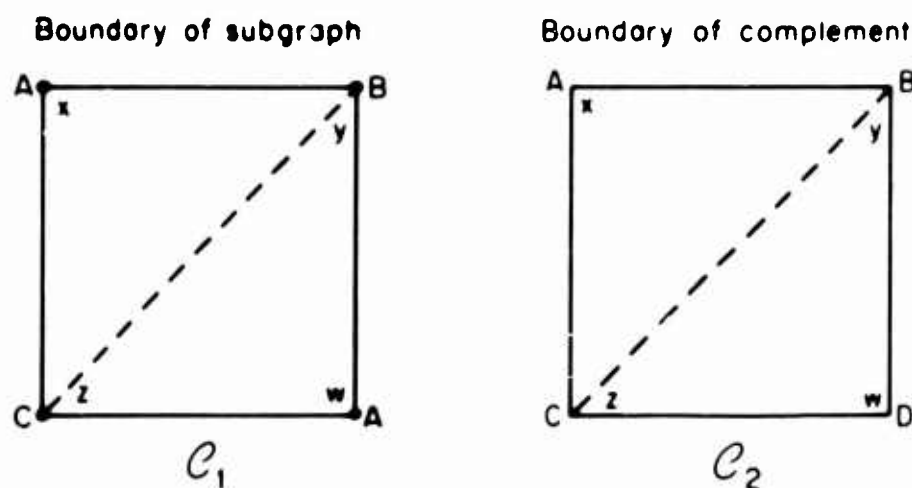Boundary of subgraph   Boundary of complement

C_1 C_2

Fig. 1

In the following it will be shown, under an inductive
hypothesis that graphs smaller than the original graph can
be four—colored, that each of the components will have
several possible colorings, and there will always be a
pair of colorings for the components that will match.

Under the inductive hypothesis, out of the four pos-
sible colorings of the boundary squares for subgraphs and
complements completed as in Fig. 1, the only two cases
that will not match after relabeling are the ones where
x and w have the same colors in one component and different
colors in the other. Assume that the situation is as shown

in Fig. 1. Now, in $\mathcal{C}_2$ the color of w can be changed to A without affecting the color of x unless there is a chain of points within the complement connecting x and w alternatively colored A, D, A, ... A, D, as shown in Fig. 2. For the color of w can be changed to A, the color of all points connected to w and colored A can be changed to D, and so on. In the case of finite graphs, this process will terminate, and, in the absence of an A − D chain between x and w, the color of x will remain unchanged. Hence, in the absence of such a chain, $\mathcal{C}_2$ can be recolored to a $\mathcal{C}_2'$, which matches $\mathcal{C}_1$.
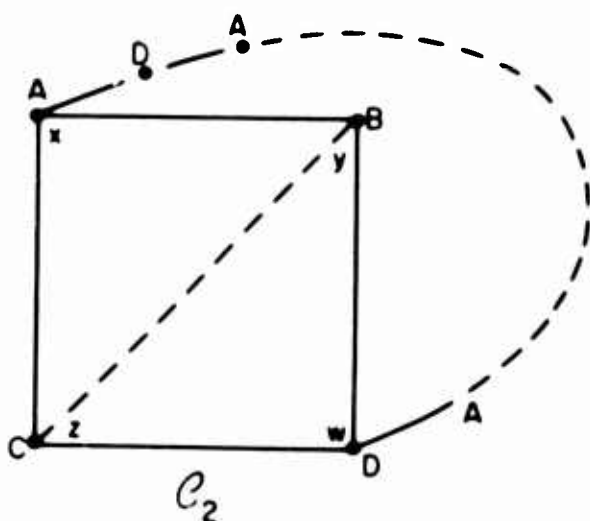

Fig. 2

In the case where there is an A − D chain between x and w, then y and z are isolated in the sense that if the added connection between y and z is removed, there cannot be a B − C chain between y and z, for it would have to intersect the A − D chain between x and w, and none of the points on

that chain are colored either B or C.   Hence when an A — D
chain between x and w exists, C_2 can be recolored to C_2''
so that y and z have the same color.   This does not produce
a match with C_1.

In this case, consider the situation with the added
connection between x and w, rather than between y and z,
for the subgraph.   (See Fig. 3.)   Under the inductive
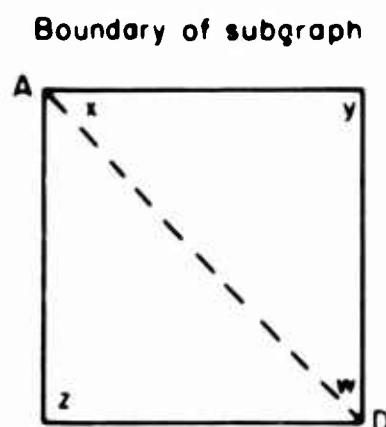
Boundary of subgraph

Fig. 3

hypothesis,   there is a coloring for the completed subgraph,
in which x and w have different colors, but the colors of
y and z are not determined.   There are two possibilities:
C_1' where y and z have different colors; and C_1'', where
y and z have the same color. C_1' matches C_2, and C_1'' matches
C_2''.

A similar argument will, of course, hold for the converse
case where the original colorings have different colors for
x and w in the subgraph and the same color for x and w in
the complement.   Thus, in all cases there is a pair of
compatible colorings.